Stat 821 Homework 1 Solution

Question 1.8

Proof: Suppose the set of median is M, $\forall m \in M, \forall a < m$

$$\begin{aligned}
& E|X-a| - E|X-m| \\
= & \int_{X\geq a}(x-1)dP + \int_{X<a}(a-X)dP - \int_{X\geq m}(X-m)dP - \int_{X<m}(m-X)dP \\
= & \int_{a\leq X<m}(2X-a-m)dP + \int_{X>m}(m-a)dP + \int_{X<a}(a-m)dP \\
\geq & \ 0
\end{aligned}$$

Last inequality holds since $P(X \geq m) \geq 1/2$, $P(X < m) \leq 1/2$ and $a < m$.

By a similar argument, we can show that $E|X-a| - E|X-m| \geq 0$ for $\forall m \in M, \forall a > m$.

If $a = m$ and $E|X-a| = E|X-m|$. $\forall m_1, m_2 \in M$ and $m_1 < m_2$,

$$P(X \geq m_2) \geq 1/2 \quad P(X \leq m_1) \geq 1/2 \quad \Rightarrow \quad P(m_1 < X < m_2) = 0$$

$$P(X \geq m_2) = P(X \leq m_1) = 1/2 \quad \Rightarrow \quad P(m_1 \leq X < m_2) = 0$$

and hence $E|X-a| - E|X-m| = 0$. Therefore, $E|X-a|$ is minimized by any median of X.

Question 1.11

(a) f_1 and f_2 are continuous symmetric densities. Thus $(f_1 - f_2)$ is a continuous symmetric function.

$$f_1(0) > f_2(0) \Leftrightarrow f_1(0) - f_2(0) > 0 \quad \text{or} \quad (f_1 - f_2)(0) > 0$$

$$\forall \epsilon > 0, \exists \delta_\epsilon > 0, \text{s.t. if } |x-0| < \delta_\epsilon, |(f_1-f_2)(x) - (f_1-f_2)(0)| < \epsilon$$

since $(f_1 - f_2)$ is continuous. Set $\epsilon = (f_1 - f_2)(0) > 0$. $\exists \delta$, s.t. if $-\delta < x < \delta$,

$$|(f_1 - f_2)(x) - (f_1 - f_2)(0)| < (f_1 - f_2)(0)$$

or

$$0 < (f_1 - f_2)(x) < 2(f_1 - f_2)(0)$$

Therefore

$$P(|\delta_1 - \theta| < \delta) - P(|\delta_2 - \theta| < \delta) = \int_{-\delta<x<\delta} (f_1 - f_2)(x)dx > 0$$

(b) $\delta_1 \sim f_1 = f(x)$, $\delta_2 \sim f_2 = 2(f * f)(2x) = \int_{-\infty}^{\infty} 2f(2x - \tau)f(\tau)d\tau$

$$P[|\delta_1| < c] > P[|\delta_2| < c] \Leftrightarrow \int_{-c}^{c} f(x)dx > \int_{-c}^{c} 2(f*f)(2x)dx = \int_{-c}^{c} 2\int_{-\infty}^{\infty} f(2x-\tau)f(\tau)d\tau dx$$

$$\begin{aligned} f_2(-x) &= 2(f * f)(-2x) \\ &= \int_{-\infty}^{\infty} 2f(-2x - \tau)f(\tau)d\tau \\ &= \int_{-\infty}^{\infty} 2f(2x - t)f(t)dt \\ &= 2(f * f)(2x) \\ &= f_2(x) \end{aligned}$$

Thus f_2 is symmetric, f is continuous $\Rightarrow$ f_2 is continuous.

$$f_2(0) = \int_{-\infty}^{\infty} 2f(-\tau)f(\tau)d\tau = 2\int_{-\infty}^{\infty} f^2(x)dx < f(0) = f_1(0)$$

Therefore, by part(a), $P(|\delta_1| < c) > P(|\delta_2| < c)$ for some c.

Question 3.8 Let $f_X = E(\lambda, 1)$, $f_Y = E(\mu, 1)$.

$$P(Z \le z, W=1) = \begin{cases} \int_\lambda^z e^{-(x-\lambda)}dx \int_x^\infty e^{-(y-\mu)}dy = \frac{1}{2}e^{\mu-\lambda} - \frac{1}{2}e^{\mu+\lambda-2z} & \text{if } z > \lambda \\ 0 & \text{if } z \le \lambda \end{cases}$$

$$P(Z \le z, W=0) = \begin{cases} 1 - \frac{1}{2}e^{\mu-\lambda} - \frac{1}{2}e^{\mu+\lambda-2z} & \text{if } z > \lambda \\ 1 - e^{\mu-z} & \text{if } \mu < z \le \lambda \\ 0 & \text{if } z \le \lambda \end{cases}$$

$$P(Z \le z) = \begin{cases} 1 - e^{\mu+\lambda-2z} & \text{if } z > \lambda \\ 1 - e^{\mu-z} & \text{if } \mu < z \le \lambda \\ 0 & \text{if } z \le \lambda \end{cases}$$

$P(W=1) = \frac{1}{2}e^{\mu-\lambda}$ $P(W=0) = 1 - \frac{1}{2}e^{\mu-\lambda}$. Thus, Z and W are dependent unless $\mu = \lambda$.

Question 5.1

(i) The natural parameter space is the set of η's s.t.

$$\int e^{\eta x}\, e^{-|x|}\, dx < \infty$$

Notice that

$$\int e^{\eta x - |x|}\, dx = \int_{x \le 0} e^{\eta x + x}\, dx + \int_{x > 0} e^{\eta x - x}\, dx$$

In order for $\int e^{\eta x}\, e^{-|x|}\, dx < \infty$, it must be true that

$$\int_{x \le 0} e^{(\eta+1)x}\, dx < \infty \;\Rightarrow\; \eta + 1 > 0 \;\Rightarrow\; \eta > -1$$

and

$$\int_{x > 0} e^{(\eta-1)x}\, dx < \infty \;\Rightarrow\; \eta - 1 > 0 \;\Rightarrow\; \eta < 1$$

Thus $\{\eta : -1 < \eta < 1\}$ is the natural parameter space.

(ii) Let

$$\int \frac{e^{\eta x}\, e^{-|x|}}{1+x^2}\, dx < \infty$$

Since

$$\frac{e^{\eta x - |x|}}{1+x^2} \le e^{\eta x - |x|} \quad \text{everywhere w.r.t. Lesbegue measure}$$

we have

$$\int \frac{e^{\eta x-|x|}}{1+x^2}\,dx \leq \int e^{\eta x-|x|}\,dx < \infty \quad \forall \eta \in (-1,1)$$

Also notice the for $\eta = 1$

$$\int \frac{e^{\eta x-|x|}}{1+x^2}\,dx = \int_{x\leq 0} \frac{e^{2x}}{1+x^2}\,dx + \int_{x>0} \frac{1}{1+x^2}\,dx < \infty$$

for $\eta = -1$

$$\int \frac{e^{\eta x-|x|}}{1+x^2}\,dx = \int_{x>0} \frac{e^{-2x}}{1+x^2}\,dx + \int_{x\leq 0} \frac{1}{1+x^2}\,dx < \infty$$

But for $\eta > 1$ notice that

$$\frac{e^{(\eta-1)x}}{1+x^2} \to \infty \quad \text{as} \quad x \to \infty$$

So for $\eta > 1$

$$\int \frac{e^{\eta x-|x|}}{1+x^2}\,dx \geq \int_{x>0} \frac{e^{(\eta-1)x}}{1+x^2}\,dx = \infty$$

Similarly for $\eta < -1$

$$\int \frac{e^{\eta x-|x|}}{1+x^2}\,dx \geq \int_{x\leq 0} \frac{e^{(\eta-1)x}}{1+x^2}\,dx = \infty$$

Thus the natural parameter space is given by $\{\eta : -1 \leq \eta \leq 1\}$.

Question 5.2

It is straightforward using the hint in the textbook.